Counting number of factorizations of a natural number

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Abstract

In this note we describe a new method of counting the number of unordered factorizations of a natural number by means of a generating function and a recurrence relation arising from it, which improves an earlier result in this direction.

1 Introduction

Consider the natural number 18. It has 4 distinct "factorizations", namely

$$18 = 2.3.3 = 2.9 = 3.6 = 18.$$

Similarly, there are 9 ways of factorizing 36: $36 = 2.2.3.3 = 2.2.9 = 2.3.6 = 3.3.4 = 2.18 = 3.12 = 4.9 = 6.6 = 36$. Our problem is to find this number for any natural number n. Since we are not distinguishing between $2 \cdot 9$ and $9 \cdot 2$, such a factorization is called *unordered*. A ***partition*** of a natural number n is a representation of n as the sum of any number of positive integral parts, where the order of the parts is irrelevant. The number of such partitions of n is known as the ***partition function*** and is denoted by $p(n)$. Likewise the function $p^*(n)$ denotes the number of ways of expressing n as a product of positive integers greater than 1, the order of the factors in the product being irrelevant. For convenience, $p^*(1)$ is assumed to be 1. Clearly $p^*(n)$ is the number of unordered factorizations of n. In 1983, Hughes and Shalit [6] obtained a bound for $p^*(n)$, namely, $p^*(n) \leqslant 2n^{\sqrt{2}}$ which was then improved to $p^*(n) \leqslant n$ by Mattics and Dodd [7] in 1986. By this time Canfield, Erdös and Pomerance [2] modified a result of Oppenheim regarding the maximal order of $p^*(n)$. They obtained another bound for $p^*(n)$ and described an algorithm for it. An average estimate for $p^*(n)$ was given by Oppenheim [8] which was also proved independently by Szekeres and Turan [9]. Finally, in 1991, Harris and Subbarao [5] came with a generating function and a recursion formula for $p^*(n)$. One may consider [1] and [3] for some problems

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associated with $p^*(n)$. For a list of values and computer programming one may consider the website: http://www.research.att.com/cgi-bin/access.cgi/as/njas/sequences/ (sequence no. A001055).

In this note, we describe a new method for counting $p^*(n)$ and obtain a generating function for it which is followed by a recurrence relation that generalizes the one given by Harris and Subbarao [5] as well as the one given by Euler for $p(n)$. The final recursion formula improves the one given in [5] as it contains less number of terms. It is important to note that we wish to develop an algebraic approach to the problem which might be helpful for other similar situations in future. Also we note some errors in describing an equivalent form of the recurrence relation in [5]. Throughout the note we denote set of all natural numbers, non-negative integers, integers and rational numbers by $\mathbb{N},\ \mathbb{Z}_0^+,\ \mathbb{Z},\ \mathbb{Q}$ respectively.

2 Representation of numbers by polynomials

Consider the monoid $(\mathbb{N}, \cdot)$ of natural numbers under usual multiplication. For any natural number n, let $S(n)$ be the submonoid of $(\mathbb{N}, \cdot)$, generated by the set of prime factors of n, i.e., if the prime factorization of n is

$$n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}, \tag{2.1}$$

where p_i are distinct primes, $n_i \in \mathbb{N}$ for all $i = 1, 2, \dots, k,\ (k \in \mathbb{N})$, then

$$S(n) = \left\{ p_1^{r_1} p_2^{r_2} \dots p_k^{r_k} \in \mathbb{N} \mid r_i \in \mathbb{Z}_0^+,\ i = 1, 2, \dots, k \right\}.$$

We show that $S(n)$ has an interesting algebraic structure. Define the partial ordering $\leq^{\cdot}$ on $S(n)$ by

$$a \leq^{\cdot} b \iff a \text{ divides } b.$$

This ordering on $S(n)$ is, in fact, a lattice ordering where $a \vee b = \operatorname{lcm}(a, b)$ and $a \wedge b = \gcd(a, b)$ for all $a, b \in S(n)$. Moreover this lattice is distributive and bounded below by 1. A monoid S is called a ***lattice-ordered semigroup*** if it has a lattice ordering that satisfies $a(b \vee c) = ab \vee ac$ and $(b \vee c)a = ba \vee ca$, for all $a, b, c \in S$. Now for all $a, b, c \in S(n)$, $a\{\operatorname{lcm}(b, c)\} = \operatorname{lcm}(ab, ac)$. So we have the following proposition:

Proposition 2.1. *For any natural number n, $(S(n), \cdot, \leq^{\cdot})$ is a lattice-ordered semigroup.*

Definition 2.2. Now corresponding to each natural number n we associate a polynomial in the polynomial semiring $\mathbb{Z}_0^+[x]$ as

$$f(x; n) = n_1 + n_2 x + n_3 x^2 + \dots + n_k x^{k-1},$$

where n has the prime factorization (2.1).

Next we define a binary relation $\leqq$ on $\mathbb{Z}_0^+[x]$ as follows:
Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$. Then

$$f(x) \leqq g(x) \iff m \leqslant n \text{ and } a_i \leqslant b_i \text{ for all } i = 0, 1, 2, \ldots, m.$$

Clearly $\leqq$ is a partial ordering on $\mathbb{Z}_0^+[x]$. Finally, let us denote the set of all polynomials in $\mathbb{Z}_0^+[x]$ of degree less than k by $P_k[x]$.

Theorem 2.3. *$(P_k[x], +, \leqq)$ is a lattice-ordered semigroup which is isomorphic to $(S(n), \cdot, \leq^{\cdot})$, where n has the prime factorization (2.1).*

Proof. Let $f(x) = a_1 + a_2x + a_3x^2 + \cdots + a_kx^{k-1}$, $g(x) = b_1 + b_2x + b_3x^2 + \cdots + b_kx^{k-1} \in P_k[x]$. Then it is routine to verify that $f \vee g = c_1 + c_2x + c_3x^2 + \cdots + c_kx^{k-1}$ and $f \wedge g = d_1 + d_2x + d_3x^2 + \cdots + d_kx^{k-1}$, where $c_i = \max\{a_i, b_i\}$ and $d_i = \min\{a_i, b_i\}$. Thus $(P_k[x], \leqq)$ is a lattice. Obviously, $(P_k[x], +)$ is an abelian monoid where the identity element is the zero polynomial. Now choose $f(x) = \sum\limits_{i=1}^{k} a_ix^{i-1}$, $g(x) = \sum\limits_{i=1}^{k} b_ix^{i-1}$, $h(x) = \sum\limits_{i=1}^{k} c_ix^{i-1} \in P_k[x]$. Then $f + (g \vee h) = \sum\limits_{i=1}^{k} \big(a_i + \max\{b_i, c_i\}\big)x^{i-1} = \sum\limits_{i=1}^{k} \big(\max\{a_i + b_i, a_i + c_i\}\big)x^{i-1} = (f + g) \vee (f + h)$. Therefore $(P_k[x], +, \leqq)$ is a lattice-ordered semigroup.

Now define a map $\psi : S(n) \longrightarrow P_k[x]$ by $\psi(m) = f(x; m)$ for all $m \in S(n)$. That ψ is bijective follows from Definition 2.2. Let $m_1 = \prod\limits_{i=1}^{k} p_i^{r_{1i}}$, $m_2 = \prod\limits_{i=1}^{k} p_i^{r_{2i}} \in S(n)$. Then

$$\begin{aligned}
& m_1 \leq^{\cdot} m_2 \\
\iff & m_1 \text{ divides } m_2 \\
\iff & r_{1i} \leqslant r_{2i} \text{ for each } i = 1, 2, \ldots, k \\
\iff & r_{11} + r_{12}x + r_{13}x^2 + \cdots + r_{1k}x^{k-1} \leqq r_{21} + r_{22}x + r_{23}x^2 + \cdots + r_{2k}x^{k-1} \\
\iff & f(x; m_1) \leqq f(x; m_2) \\
\iff & \psi(m_1) \leqq \psi(m_2).
\end{aligned}$$

Also $m_1m_2 = \prod\limits_{i=1}^{k} p_i^{r_{1i}+r_{2i}}$. Then $f(x; m_1m_2) = \sum\limits_{i=1}^{k}(r_{1i} + r_{2i})x^{i-1} = \sum\limits_{i=1}^{k} r_{1i}x^{i-1} + \sum\limits_{i=1}^{k} r_{2i}x^{i-1} = f(x; m_1) + f(x; m_2)$. Thus $\psi(m_1m_2) = \psi(m_1) + \psi(m_2)$. Therefore ψ is an isomorphism, i.e., $(S(n), \cdot, \leq^{\cdot}) \cong (P_k[x], +, \leqq)$, as required. $\square$

Definition 2.4. For any $f(x) \in \mathbb{Z}_0^+[x]$, let $p(f)$ denote the number of partitions of the polynomial $f(x)$ in terms of addition of polynomials (not all distinct) in $\mathbb{Z}_0^+[x]$, where the order of addition is irrelevant. We assume $p(0) = 1$.

For example, the distinct partitions of the polynomial $2+x$ in $\mathbb{Z}_0^+[x]$ are

$$\begin{aligned} 2+x &= (1)+(1)+(x) \\ &= (1)+(1+x) \\ &= (2)+(x) \\ &= (2+x) \end{aligned}$$

Note that $f(x;12) = 2+x$ and compare the above partitions with usual factorizations: $12 = 2.2.3 = 2.(2.3) = 2^2.3 = 12$.

Theorem 2.5. *For any natural number n, $p^*(n) = p(f(x;n))$.*

Proof. Let $n \in \mathbb{N}$ and $F(n)$ denote the set of all factors of n. Then $(F(n), \dot{\leq})$ is a sublattice of $S(n)$. On the other hand, define[1] the set

$$S(f(x)) = \left\{ g(x) \in \mathbb{Z}_0^+[x] \mid g(x) \leqq f(x) \right\}.$$

Then $S(f(x))$ is a sublattice of $P_k[x]$, where $f(x) = f(x;n)$ and n has the prime factorization (2.1). By Theorem 2.3, it follows that the restriction of the map ψ on $F(n)$ is a lattice isomorphism from $(F(n), \dot{\leq})$ onto $\big(S(f(x;n)), \leqq\big)$. Indeed, let $m = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k} \in F(n)$ for some $r_1, r_2, \dots, r_k \in \mathbb{Z}_0^+$. Since m is a factor of n, we have $r_i \leqslant n_i$ for each $i = 1, 2, \dots, k$. Hence $\psi(m) = f(x;m) = r_1 + r_2 x + r_3 x^2 + \cdots + r_k x^{k-1} \leqq n_1 + n_2 x + n_3 x^2 + \cdots + n_k x^{k-1} = f(x;n)$ which implies $\psi(m) \in S(f(x;n))$. Conversely, let $g(x) \in S(f(x;n))$. Then $\deg g(x) \leqslant \deg f(x) = k-1$. Let $g(x) = b_1 + b_2 x + b_3 x^2 + \cdots + b_k x^{k-1}$ for some $b_1, b_2, \dots, b_k \in \mathbb{Z}_0^+$. Then $b_i \leqslant n_i$ for each $i = 1, 2, \dots, k$ and $g(x) = g(x; p_1^{b_1} p_2^{b_2} \dots p_k^{b_k}) = \psi(p_1^{b_1} p_2^{b_2} \dots p_k^{b_k})$, which implies $\psi(F(n)) = S(f(x;n))$. Thus we have $(F(n), \dot{\leq}) \cong \big(S(f(x;n)), \leqq\big)$. Also since $f(x; m_1 m_2) = f(x;m_1) + f(x;m_2)$ for all $m_1, m_2 \in F(n)$, there exists a one-to-one correspondence between factorizations $n = m_1 m_2 \dots m_r$ of n with partitions $f(x;m_1) + f(x;m_2) + \cdots + f(x;m_r) = f(x; m_1 m_2 \dots m_r) = f(x;n)$ of $f(x;n)$. Thus we have $p^*(n) = p(f(x;n))$. $\square$

Corollary 2.6. *Let p be a prime number and $n \in \mathbb{N}$. Then $p^*(p^n) = p(n)$.*

Remark 2.7. It is clear that the value of $p^*(n)$ is independent of the particular primes involved in the prime factorization expression of n, i.e., if n has the prime factorization (2.1) and $m = q_1^{n_1} q_2^{n_2} \dots q_k^{n_k}$, where q_i are distinct primes, then $p^*(m) = p^*(n)$. Thus the polynomial $f(x;n)$ in Definition 2.2 may be different for different arrangement of primes in the prime factorization

[1] $S(f(x))$ is called the ***section*** of $f(x)$ in $\mathbb{Z}_0^+[x]$.

expression of n. But the value of $p(f(x;n))$ remains same for each such arrangements. In particular, $p(2+x) = p^*(2^2.3) = 4 = p^*(2.3^2) = p(1+2x)$. Indeed, the distinct partitions of the polynomial $1+2x$ in $\mathbb{Z}_0^+[x]$ are

$$\begin{aligned} 1+2x &= (1)+(x)+(x) \\ &= (1+x)+(x) \\ &= (1)+(2x) \\ &= (1+2x) \end{aligned}$$

More generally, $p^*(p^2q) = 4$ for any pair of distinct primes p, q.

3 Generating function and recurrence relations

Let n be a natural number. We know that the number of partitions, $p(n)$ of n is given [4] by the following classical generating function found by Euler:

$$\begin{aligned} F(x) &= \frac{1}{(1-x)(1-x^2)(1-x^3)\dots} && (3.1) \\ &= \prod_{n=1}^{\infty} \frac{1}{1-x^n} && (3.2) \\ &= \prod_{i=1}^{\infty} \left(1 + \sum_{n=1}^{\infty} x^{ni}\right) && (3.3) \\ &= 1 + \sum_{n=1}^{\infty} p(n)x^n. && (3.4) \end{aligned}$$

In the above equalities, since (3.3) provides all possible positive integral powers of x less than n (with all possible multiplicities), product of these terms produce the term x^n as many times as n can be expressed as a sum of positive integers which is exactly the number of partitions of n, i.e., the term x^n occurs $p(n)$ times. So we get the coefficient $p(n)$ of x^n in (3.4). Similarly, if we wish to find the number of partitions of the polynomial $f(x)$ in the polynomial semiring $\mathbb{Z}_0^+[x]$, we have to consider the product of summations which provides all possible polynomials in $\mathbb{Z}_0^+[x]$ less than $f(x)$ (with all possible multiplicities) as indices. Thus we have the following *formal* generating function for $p(f(x))$:

$$\begin{aligned} \mathcal{F}(x) &= \prod_{g \in \mathbb{Z}_0^+[x]^\star} \frac{1}{1-e^{g(x)}} && (3.5) \\ &= \prod_{g \in \mathbb{Z}_0^+[x]^\star} \left(1 + \sum_{n=1}^{\infty} e^{ng(x)}\right) && (3.6) \\ &= 1 + \sum_{f \in \mathbb{Z}_0^+[x]^\star} p(f)\ e^{f(x)}, && (3.7) \end{aligned}$$

where $\mathbb{Z}_0^+[x]^\star = \mathbb{Z}_0^+[x] \setminus \{0\}$.

Remark 3.1. The expressions (3.5)-(3.7) are merely formal in the sense that for any particular $f(x) \in \mathbb{Z}_0^+[x]$, the coefficients of $e^{f(x)}$ in either side are same. So the convergence problem does not arise here. However, if one insists on it, one may replace e by $e_1 = \frac{1}{e}$ in which case (3.5)-(3.7) are absolutely and uniformly convergent for all positive integral values of x. For example, consider $\mathcal{F}(1) = \prod_{g \in \mathbb{Z}_0^+[x]^\star} \frac{1}{1-e_1^{g(1)}}$. The product $\prod_{g \in \mathbb{Z}_0^+[x]^\star} (1-e_1^{g(1)})$ is convergent if $\sum_{g \in \mathbb{Z}_0^+[x]^\star} e_1^{g(1)}$ is convergent. Now $\sum_{g \in \mathbb{Z}_0^+[x]^\star} e_1^{g(1)} = \sum_{n=1}^{\infty} p(n) e_1^n$ which is absolutely and uniformly convergent for $e_1 = \frac{1}{e}$ [4].

Now (3.5) can be written in the form:

$$\mathcal{F}(x) = \prod_{\substack{g \in \mathbb{Z}_0^+[x]^\star \\ g \text{ is primitive}}} \frac{1}{\prod_{n=1}^{\infty} (1 - e^{ng(x)})} \tag{3.8}$$

which is again by (3.4),

$$\mathcal{F}(x) = \prod_{\substack{g \in \mathbb{Z}_0^+[x]^\star \\ g \text{ is primitive}}} \Big(1 + \sum_{n=1}^{\infty} p(n)\ e^{ng(x)}\Big) \tag{3.9}$$

So we have the following generating function for $p(f(x))$:

$$1 + \sum_{f \in \mathbb{Z}_0^+[x]^\star} p(f)\ e^{f(x)} = \prod_{\substack{g \in \mathbb{Z}_0^+[x]^\star \\ g \text{ is primitive}}} \Big(1 + \sum_{n=1}^{\infty} p(n)\ e^{ng(x)}\Big) \tag{3.10}$$

Using this we describe a method of calculating $p(f(x))$:

$$p(f(x)) = \text{ coefficient of } e^{f(x)} \text{ in } \prod_{\substack{0 < g \leqslant f,\ g \in \mathbb{Z}_0^+[x] \\ g \text{ is primitive}}} \Big(1 + \sum_{n=1}^{\infty} p(n)\ e^{ng(x)}\Big). \tag{3.11}$$

Example 3.2. Let $n = 12$. Then $n = 2^2.3$ and so the associated polynomial $f(x; 12) = 2 + x$. Now primitive polynomials less than or equal to $2 + x$ in $\mathbb{Z}_0^+[x]$ are $1, x, 1 + x$ and $2 + x$. So we have

$$\begin{aligned}
p(f(x;12)) &= p(2+x) \\
&= \text{ coefficient of } e^{2+x} \text{ in} \\
&\quad \big(1 + p(1)e + p(2)e^2\big)\ \big(1 + p(1)e^x\big)\ \big(1 + p(1)e^{1+x}\big)\ \big(1 + p(1)e^{2+x}\big) \\
&= \text{ coefficient of } e^{2+x} \text{ in } (1 + e + 2e^2)(1 + e^x)(1 + e^{1+x})(1 + e^{2+x}) \\
&= \text{ coefficient of } e^{2+x} \text{ in } (1 + e + 2e^2)(1 + e^x + e^{1+x} + e^{2+x}) \\
&= \text{ coefficient of } e^{2+x} \text{ in } 1 + e + 2e^2 + e^x + 2e^{1+x} + 4e^{2+x} \\
&= 4.
\end{aligned}$$

Thus we get that $p^*(12) = p(f(x;12)) = 4$.

Remark 3.3. **(i)** Note that in each step of the above calculation, we omit the terms $e^{h(x)}$ whenever $h(x) > f(x;12)$, as these terms have no further contribution in forming the term $e^{f(x;12)}$.

(ii) By the process of calculating $p(f(x))$, we are getting all the values of $p(g(x))$ for all $g(x) \leqslant f(x)$ in $\mathbb{Z}_0^+[x]$. For example, the above calculation gives us

$$p(1) = 1,\ p(2) = 2,\ p(x) = 1,\ p(1+x) = 2.$$

Next we wish to obtain a recurrence relation for $p(f(x))$. From (3.5) and (3.7), we get that

$$\Big(1 + \sum_{f \in \mathbb{Z}_0^+[x]^\star} p(f)\ e^{f(x)}\Big) \cdot \prod_{g \in \mathbb{Z}_0^+[x]^\star} \Big(1 - e^{g(x)}\Big) \ = \ 1. \tag{3.12}$$

Now taking formal derivatives[2] on both sides of (3.12) we get,

$$\Big(\sum_{f \in \mathbb{Z}_0^+[x]^\star} p(f)\ e^{f(x)} \cdot f'(x)\Big) \cdot \prod_{g \in \mathbb{Z}_0^+[x]^\star} (1 - e^{g(x)}) \ +$$
$$\Big(1 + \sum_{f \in \mathbb{Z}_0^+[x]^\star} p(f)\ e^{f(x)}\Big) \cdot \Big(\sum_{g \in \mathbb{Z}_0^+[x]^\star} \Big\{ \big(-e^{g(x)} g'(x)\big) \prod_{\substack{g_1 \in \mathbb{Z}_0^+[x]^\star \\ g_1 \neq g}} \big(1 - e^{g_1(x)}\big) \Big\} \Big) \ = \ 0$$

which implies

$$\sum_{f \in \mathbb{Z}_0^+[x]^\star} p(f)\ e^{f(x)} \cdot f'(x) \ = \ \Big(1 + \sum_{f \in \mathbb{Z}_0^+[x]^\star} p(f)\ e^{f(x)}\Big) \cdot \Big(\sum_{g \in \mathbb{Z}_0^+[x]^\star} \Big\{ e^{g(x)} g'(x) \cdot \big(1 + \sum_{r=1}^{\infty} e^{rg(x)}\big) \Big\} \Big). \tag{3.13}$$

Then equating the coefficient of $e^{f(x)}$ of both sides we get

$$p(f)\ f'(x) \ = \ \Big(\sum_{r | c(f)} \frac{1}{r} \Big) \cdot f'(x) \ + \sum_{\substack{0<g<f \\ g \in \mathbb{Z}_0^+[x]}} g'(x) \Big(\sum_{r=1}^{k_g} p(f - rg) \Big). \tag{3.14}$$

where $c(f)$ is the content[3] of the polynomial $f(x)$ and $k_g = \max\{r \in \mathbb{N} \mid f(x) > rg(x)\}$. Considering $p(0) = 1$ and replacing rg by g we finally have

$$p(f)\ f'(x) \ = \ \sum_{\substack{0<g \leqq f \\ g \in \mathbb{Z}_0^+[x]}} \lambda(g)\ g'(x)\ p(f - g). \tag{3.15}$$

where $\lambda(g) = \sum_{i | c(g)} \frac{1}{i}$. Now since polynomials on both sides of (3.15) are identical, we may equate coefficients of each power of x. So if $f(x) = n_1 + n_2 x + n_3 x^2 + \cdots + n_k x^{k-1}$ and $b_2(g)$ denotes the

[2]The ***formal derivative*** of a polynomial $h(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k \in \mathbb{Z}_0^+[x]$ is defined by $h'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + k a_k x^{k-1}$. One can easily extend this definition for formal power series. The operator derivative is additive and follows Leibnitz' rule. It is routine to verify that the derivative of $e^{h(x)} = e^{h(x)} h'(x)$.

[3]i.e., gcd of all coefficients of the polynomial $f(x)$.

coefficient of x in $g(x)$ for each $g \in \mathbb{Z}_0^+[x]$ such that $0 < g(x) \leqq f(x)$, then equating constant terms in (3.15) we have

$$p(f)\ n_2 \ = \sum_{\substack{0<g\leqq f \\ g\in \mathbb{Z}_0^+[x]}} \lambda(g)\ b_2(g)\ p(f-g). \tag{3.16}$$

Also by Remark 2.7, one may rearrange coefficients of $f(x)$ which does not change the value of $p(f)$. Thus for each $i = 1, 2, \ldots, k$, we may rearrange coefficients of $f(x)$ in such a way that n_i will be the coefficient of x. Then we have

$$p(f)\ n_i \ = \sum_{\substack{0<g\leqq f \\ g\in \mathbb{Z}_0^+[x]}} \lambda(g)\ b_i(g)\ p(f-g), \tag{3.17}$$

where $b_i(g)$ denotes the coefficient of x^{i-1} in $g(x)$ for each $g \in \mathbb{Z}_0^+[x]$ such that $0 < g(x) \leqq f(x)$. Therefore suitably multiplying the above relations by powers of x and adding we get

$$p(f)\ f(x) \ = \sum_{\substack{0<g\leqq f \\ g\in \mathbb{Z}_0^+[x]}} \lambda(g)\ g(x)\ p(f-g), \tag{3.18}$$

Remark 3.4. **(i)** We first note that (3.18) is a nice generalization of Euler's recurrence relation for $p(n)$, $(n \in \mathbb{N})$ which is given by

$$n\, p(n) = \sum_{j=1}^{n} \sigma(j)\, p(n-j), \tag{3.19}$$

where $\sigma(j)$ denotes the sum of all divisors of j and $p(0)$ is assumed to be 1. Now (3.19) is immediately obtained from (3.18) by putting $f(x) = n$ (the constant polynomial), as we already have, by Corollary 2.6, $p(n) = p^*(p^n)$. Note that $\lambda(j)j = \sigma(j)$ for all $j = 1, 2, \ldots, n$.

(ii) The recurrence relation (3.18) also generalizes the one obtained by Harris and Subbarao [5]. In Remark 2 of [5] (pg.477), they described an equivalent form of their recurrence relation as follows:

Consider the vector $\vec{\alpha}(n) = (\alpha_1, \alpha_2, \ldots \alpha_k)$ for the natural number $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k}$. Then

$$p^*(\vec{\alpha})\ \|\vec{\alpha}\| \ = \sum_{\vec{0}\leq\vec{\beta}\leq\vec{\alpha}} p^*(\vec{\alpha}-\vec{\beta})\lambda(\vec{\beta})\|\vec{\beta}\|, \tag{3.20}$$

where $p^*(\vec{\alpha}) = p^*(n)$, $\|\vec{\alpha}\| = \prod_{j=1}^{k} \alpha_j$, $\lambda(\vec{\alpha}) = \sum_{i:i\big|\alpha_j \text{ for } 1\leq j\leq k\}} 1/i$ and $\vec{\beta} < \vec{\alpha}$ means that $\beta_j \leq \alpha_j$ for $1 \leq j \leq k$.

We first note that the limit under sum in (3.20) should be $\vec{0} < \vec{\beta} \leq \vec{\alpha}$ as $\lambda(\vec{0})$ is not defined. Secondly, $\|\vec{\alpha}\|$ will be $\sum_{j=1}^{k} \alpha_j$, as one may verify $p^*(18) = 5$ by (3.20), which is wrong. Finally,

while defining the ordering for vectors one has to use $\leq$ instead of $<$. However keeping aside these printing mistakes, the correct version of (3.20) is given by

$$p^*(\vec{\alpha})\ \|\vec{\alpha}\| \ = \sum_{\vec{0}<\vec{\beta}\leq\vec{\alpha}} p^*(\vec{\alpha}-\vec{\beta})\lambda(\vec{\beta})\|\vec{\beta}\|, \tag{3.21}$$

where $\|\vec{\alpha}\| = \sum_{j=1}^{k} \alpha_j$. Now (3.21) immediately follows from (3.18) by putting $x = 1$ (or, adding equations (3.17) for $i = 1, 2, \ldots, k$).

Now for any $n \in \mathbb{N}$ with prime factorization (2.1), we define $c(n) = \gcd\{n_1, n_2, \ldots n_k\}$ and $\lambda(n) = \sum_{i | c(n)} \frac{1}{i}$. Then equating constant terms in (3.18) we get

$$p(f)\ n_1 \ = \sum_{\substack{0<g\leqq f \\ g\in \mathbb{Z}_0^+[x]}} \lambda(g)\ b_1(g)\ p(f-g), \tag{3.22}$$

where $b_1(g)$ denotes the constant term of the polynomial $g(x)$ for each $g \in \mathbb{Z}_0^+[x]$ such that $0 < g(x) \leqq f(x)$. This implies

$$p^*(n)\ n_1 \ = \sum_{d | n,\ p_1 | d} r_1 \lambda(d)\ p^*\big(\frac{n}{d}\big), \tag{3.23}$$

where the summation runs over $d = p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}$ for $0 < r_1 \leqslant n_1$ and $0 \leqslant r_j \leqslant n_j,\ j = 2, 3, \ldots, k$. Thus we have

$$p^*(n)\ n_1 \ = \sum_{i=1}^{n_1} i \cdot \Big(\sum_{d | \frac{n}{p_1^{n_1}}} \lambda(p_1^i d)\ p^*\big(\frac{n}{p_1^i d}\big)\Big). \tag{3.24}$$

Remark 3.5. We note that (3.24) improves the recurrence relation obtained by Harris and Subbarao [5] as it contains less terms. In fact, the number of terms in (3.24) is $(n_2+1)(n_3+1)\ldots(n_k+1)$ less than that of (3.21) for $n = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$. Thus, in view of Remark 2.7, it is advisable to arrange prime factors of n in such a way that n_1 should be minimum among all n_i's for quicker computation.

We summarize the above results in the following:

Theorem 3.6. *Let $n = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$ be the prime factorization of a natural number n, where p_i are distinct primes, $n_i \in \mathbb{N}$ for all $i = 1, 2, \ldots, k$, $(k \in \mathbb{N})$, with $n_1 = \min\{n_i \mid i = 1, 2, \ldots, k\}$. Then*

$$p(f)\ f(x;n) \ = \sum_{\substack{0<g\leqq f \\ g\in \mathbb{Z}_0^+[x]}} \lambda(g)\ g(x)\ p(f-g), \tag{3.25}$$

where $f(x;n) = n_1 + n_2 x + n_3 x^2 + \cdots + n_k x^{k-1},\ \lambda(g) = \sum_{i \mid c(g)} \frac{1}{i},\ c(g)$ *being the content of the polynomial* $g(x)$ *for each* $g \in \mathbb{Z}_0^+[x]$ *such that* $0 < g(x) \leqq f(x)$*. In particular,*

$$p^*(n)\ n_1 \ = \ \sum_{i=1}^{n_1} i \cdot \Big(\sum_{d \mid \frac{n}{p_1^{n_1}}} \lambda(p_1^i d)\ p^*\big(\frac{n}{p_1^i d}\big) \Big),$$

where $\lambda(m) = \sum_{i \mid c(m)} \frac{1}{i},\ c(m) = \gcd\{i, r_2, \ldots r_k\}$ *for* $m = p_1^i p_2^{r_2} \ldots p_k^{r_k},\ 0 < i \leqslant n_1$ *and* $0 \leqslant r_j \leqslant n_j,\ j = 2, 3, \ldots, k$*.*

A simple formula is obtained the case of $n_1 = 1$. In that case, $c(m) = 1$ for all $m \mid n$ with $p_1 \mid m$ and So $\lambda(m) = 1$ for all such m. Thus we have

Corollary 3.7. *Let* n *be a natural number and* p *be a prime number such that* $p \nmid n$*. Then*

$$p^*(np) = \sum_{d \mid n} p^*(d). \tag{3.26}$$

In particular, for any two distinct prime numbers p, q *and for any natural number* n*,*

$$p^*(pq^n) = \sum_{i=0}^{n} p(i), \tag{3.27}$$

Example 3.8. Let $n = 72$. Then $n = 2^3.3^2 = 3^2.2^3$. Now the divisors of 2^3 are $1, 2, 4, 8$. So we have

$$\begin{aligned} & p^*(72) \cdot 2 \\ = \ & \sum_{i=1}^{2} i \cdot \Big(\sum_{d \mid 8} \lambda(3^i d)\ p^*\big(\frac{72}{3^i d}\big) \Big), \\ = \ & \big\{ p^*(24) + p^*(12) + p^*(6) + p^*(3) \big\} + \\ & \qquad 2 \cdot \big\{ \lambda(3^2)\ p^*(8) + \lambda(3^2 \cdot 2)\ p^*(4) + \lambda(3^2 \cdot 2^2)\ p^*(2) + \lambda(3^2 \cdot 2^3)\ p^*(1) \big\}. \end{aligned}$$

Now $p^*(2) = p^*(3) = 1,\ p^*(4) = p^*(2^2) = p(2) = 2,\ p^*(8) = p^*(2^3) = p(3) = 3$ by Corollary 2.6. Again $p^*(6) = p^*(3 \cdot 2) = 1 + p(1) = 2,\ p^*(12) = p^*(3 \cdot 2^2) = 2 + p(2) = 4,\ p^*(24) = p^*(3 \cdot 2^3) = 4 + p(3) = 7$ by (3.27). Thus we have

$$p^*(72) \cdot 2 = \big\{ 7 + 4 + 2 + 1 \big\} + 2 \cdot \big\{ (1 + \frac{1}{2}) \cdot 3 + 1 \cdot 2 + (1 + \frac{1}{2}) \cdot 1 + 1 \cdot 1 \big\} = 14 + 18 = 32.$$

Hence $p^*(72) = 16$. Indeed 16 factorizations of 72 are

$$\begin{aligned} 72 \ &= \ 2.2.2.3.3 \ = \ 2.2.2.9 \ = \ 2.2.3.6 \ = \ 2.2.18 \ = \ 2.3.3.4 \ = \ 2.3.12 \ = \ 2.4.9 \\ &= \ 2.6.6 \ = \ 2.36 \ = \ 3.3.8 \ = \ 3.4.6 \ = \ 3.24 \ = \ 4.18 \ = \ 6.12 \ = \ 8.9 \ = \ 72. \end{aligned}$$

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